MATH 2050C Lecture 5 (Jan 25)

Announcements:

- Thursdays classroom change SC LG23 → LSK LT3
- Quiz suspended until face-to-face teaching resumed
- Tutorial classwork submit via Blackboard

Last week ... we talked about

Completeness Property: Every $\emptyset \neq S \subseteq \mathbb{R}$ which is bounded above must have a supremum in $\mathbb{R}$.

Recall: "Archimedean Property"

- $\mathbb{N} \subseteq \mathbb{R}$ is NOT bdd above
- $\forall\, t > 0,\ \exists\, n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < t$
- $\forall\, y > 0,\ \exists!\, n \in \mathbb{N}$ s.t. $n-1 \leq y < n$

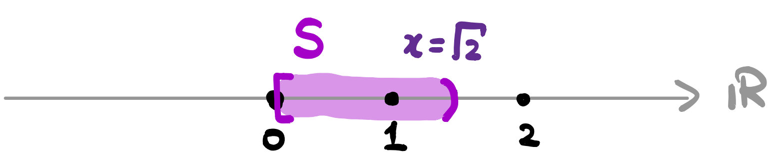
From Lecture 1, we proved that "$\sqrt{2} \notin \mathbb{Q}$", i.e.

"$\nexists\, q \in \mathbb{Q}$ s.t. $q^2 = 2$"

Thm: (Existence of $\sqrt{2}$ in $\mathbb{R}$)

$$\exists\, x \in \mathbb{R} \text{ s.t. } x^2 = 2 \text{ and } x > 0$$

Picture:



Proof: Let $S := \{ s \in \mathbb{R} : s \geq 0 , s^2 < 2 \}$

Note: $0 \in S \Rightarrow S \neq \phi$

Claim: S is bdd above.

$\forall s \in S$, we have $s^2 < 2 < 4 = 2^2 \Rightarrow s < 2$

i.e. 2 is an upper bd for S.

Apply Completeness of $\mathbb{R}$ to the subset S.

$$x := \sup S \in \mathbb{R} \quad \text{exists.}$$

Note: To see $x > 0$, we observe that $1 \in S$

$0 < 1 \leq x$ since $x := \sup S$ is an upper bd of S

It remains to show $\boxed{x^2 = 2}$

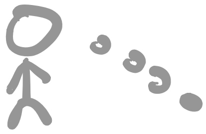
From the "trichotomy" (O2), we have to rule out

$$x^2 < 2 \qquad \text{or} \qquad x^2 > 2$$

Case 1: Suppose $x^2 < 2$.

Claim: $\exists\, n \in \mathbb{N}$ s.t. $x + \frac{1}{n} \in S$, i.e. $(x + \frac{1}{n})^2 < 2$.

This will contradict the fact that x is an upper bd. for S.



$$\left(x+\frac{1}{n}\right)^2 < 2$$
$$\Updownarrow$$
$$x^2+\frac{2x}{n}+\frac{1}{n^2} < 2$$
$$\Uparrow$$
$$x^2+\frac{2x}{n}+\frac{1}{n} < 2$$
$$\Updownarrow$$
$$\frac{2x+1}{n} < 2-x^2$$
$$\Updownarrow \text{ } \because 2x+1>0$$
$$\frac{1}{n} < \frac{2-x^2}{2x+1}$$

Pf of Claim:

By assumption of Case 1,

$$x^2 < 2 \implies 2-x^2 > 0$$

Also, $x > 0 \implies 2x+1 > 0$

Thus, $\dfrac{2-x^2}{2x+1} > 0$.

By Archimedean Property,

$\exists\, n \in \mathbb{N}$ s.t. $\underbrace{0 < \frac{1}{n} < \frac{2-x^2}{2x+1}}_{(*)}$

For this choice of $n \in \mathbb{N}$,

$$\left(x+\frac{1}{n}\right)^2 = x^2+\frac{2x}{n}+\frac{1}{n^2}$$

$\left(\because \frac{1}{n^2} \le \frac{1}{n} \text{ for all } n \in \mathbb{N}\right)$

$$\le x^2+\frac{2x}{n}+\frac{1}{n}$$

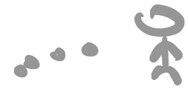
$$= x^2+\frac{2x+1}{n} \underset{\text{by } (*)}{<} 2$$

$\square$

Case 2: Suppose $x^2 > 2$.

Claim: $\exists\, m \in \mathbb{N}$ s.t. $x - \frac{1}{m}$ is still an upper bd of S

This contradicts $x := \sup S$ is the least upper bd.



Pf of Claim:

Since $x > 0$ and $x^2 - 2 > 0$,

by Archimedean Property,

$\exists\, m \in \mathbb{N}$ s.t.

$$0 < \frac{1}{m} < \frac{x^2-2}{2x} \quad \cdots\cdots (**)$$

For any $s \in S$,

$$\left(x - \frac{1}{m}\right)^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2}$$

$$> x^2 - \frac{2x}{m}$$

by $(**)$

$$> 2 > s^2$$

$x - \frac{1}{m} \geq s,\ \forall s \in S$

$\left(x - \frac{1}{m}\right)^2 \geq s^2,\ \forall s \in S$

$\left(x - \frac{1}{m}\right)^2 \geq 2 > s^2,\ \forall s \in S$

$x^2 - \frac{2x}{m} + \frac{1}{m^2} \geq 2$

$x^2 - \frac{2x}{m} \geq 2$

$\frac{x^2-2}{2x} \geq \frac{1}{m}$

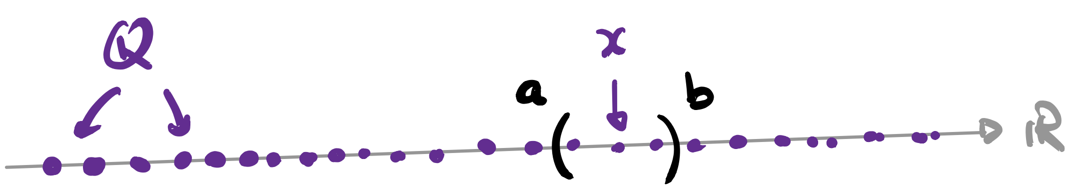
$\square$

Thm: (Density of $\mathbb{Q}$ in $\mathbb{R}$)

For any $a, b \in \mathbb{R}$ s.t. $a < b$,

$\exists\, x \in \mathbb{Q}$ s.t. $a < x < b$

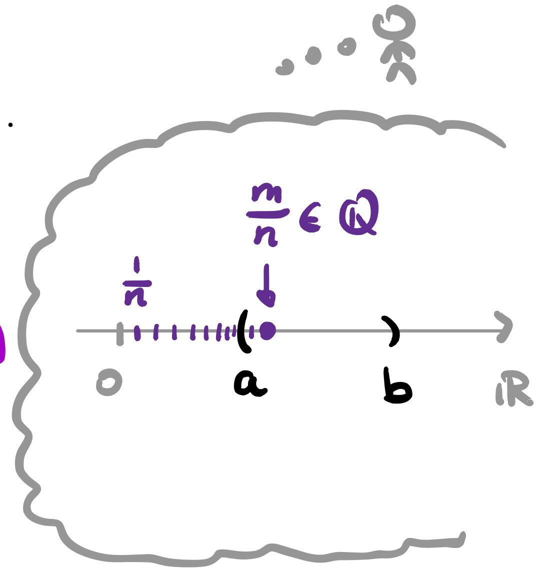
Picture:



Proof: Since $a<b$, so $b-a>0$.

By Archimedean Property.

$\exists\, n \in \mathbb{N}$ s.t. $\boxed{0 < \frac{1}{n} < b-a}$. ... (#)



By Archimedean Property.

$m-1 \le na < m$ for some $m \in \mathbb{N}$

i.e. $\frac{m}{n} - \frac{1}{n} \le a < \frac{m}{n} =: x \in \mathbb{Q}$

Claim: $\frac{m}{n} < b$

Pf of Claim:

$$\frac{m}{n} \le a + \frac{1}{n} \overset{\text{by (\#)}}{<} a + (b-a) = b$$

$\blacksquare$

Cor: The irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Pf: Take any $a<b$ in $\mathbb{R}$. Consider $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$.

by density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists\, q \in \mathbb{Q}$ s.t.

$$\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}} \implies a < \underbrace{q \cdot \sqrt{2}}_{\in\, \mathbb{R} \setminus \mathbb{Q}} < b$$

$\blacksquare$

§ Intervals (Textbook § 2.5)

$\exists$ 9 types of intervals of $\mathbb{R}$, depending on whether it is closed / open, bdd / unbdd.

<u>Notations</u>: Fix $a, b \in \mathbb{R}$, $a < b$,

$(a,b) := \{x \in \mathbb{R} : a < x < b\}$

$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}$

$(a,b] := \{x \in \mathbb{R} : a < x \leq b\}$

$[a,b) := \{x \in \mathbb{R} : a \leq x < b\}$

"bdd intervals"

$(a,\infty) := \{x \in \mathbb{R} : a < x\}$

$[a,\infty) := \{x \in \mathbb{R} : a \leq x\}$

$(-\infty, b) := \{x \in \mathbb{R} : x < b\}$

$(-\infty, b] := \{x \in \mathbb{R} : x \leq b\}$

$(-\infty, \infty) := \mathbb{R}$

"unbdd intervals"

<u>Def$^{\underline{n}}$</u>: $\text{Length}(I) := b - a > 0$

<u>Q</u>: When is $S \subseteq \mathbb{R}$ an interval?

<u>A</u>: "connectedness".

Thm: (Characterization of intervals)

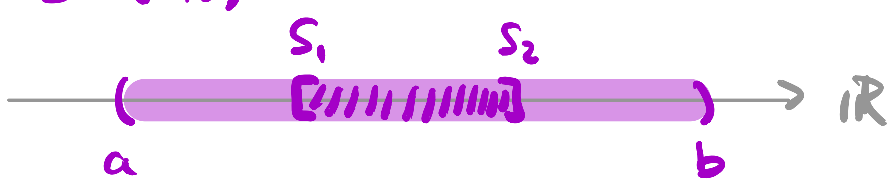
Let $S \subseteq \mathbb{R}$. Suppose:

(i) $\exists\, s_1, s_2 \in S$ s.t. $s_1 \neq s_2$

(ii) If $x, y \in S$ where $x < y$, then $[x, y] \subseteq S$

THEN: S is an interval.

Picture: $S = (a,b)$



Proof: We just treat the case S is bdd.

By completeness, $a := \inf S$, $b := \sup S$ exist.

Claim: $(a, b) \subseteq S$ $\quad(\Rightarrow$ DONE$)$

Pf of Claim: Take any $x \in (a,b)$, want $x \in S$.

- $x > a := \inf S \implies x$ cannot be a lower bd of S

  i.e. $\exists\, s' \in S$ s.t. $s' < x$

- $x < b := \sup S \implies x$ cannot be a upper bd of S

  i.e. $\exists\, s'' \in S$ s.t. $s'' > x$

But $x \in (s', s'') \subseteq [s', s''] \overset{(ii)}{\subseteq} S$, so $x \in S$.

$\square$